CONSOLIDATION PROBLEMS

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Introduction

The analysis of deformation in soils is of paramount importance in geotechnical engineering. For a long time the complex behaviour of natural deposits defied the ingenuity of engineers. The time has come that, with the aid of computers, numerical methods will allow the solution of every problem if the material law can be specified with a certain accuracy.

Boundary techniques (B.E.) have recently exploded in a splendid flowering of methods and applications that compare advantageously with other well-established procedures like the finite element method (F.E.). Its application to soil mechanics problems (Brebbia 1981) has started and will grow in the future.

This paper tries to present a simple formulation to a classical problem. In fact there is already a large amount of applications of B.E. to diffusion problems (Rizzo et al, Shaw, Chang et al, Combescure et al, Wrobel et al, Roures et al, Onishi et al) and very recently the first specific application to consolidation problems has been published by Onishi et al.

Here we develop an alternative formulation to that presented in the last reference. Fundamentally the idea is to introduce a finite difference discretization in the time domain in order to use the fundamental solution of a Helmholtz type equation governing the neutral pressure distribution.

Although this procedure seems to have been unappreciated in the previous technical literature it is nevertheless effective and straightforward to implement. Indeed for the special problem in study it is perfectly suited, because a step by step interaction between the elastic and the flow problems is needed. It allows also the introduction of non-linear elastic properties and time dependent conditions very easily as will be shown and compares well with performances of other approaches.

Basic equations

As is well known the classical consolidation problems describe the coupling of fluid and solid phases of saturated soils.

If the compressibility of solid particles is not taken into account the law of mass conservation can be written as

$$\dot{\theta} - \frac{n}{k_w}\dot{\phi} + \mathrm{div}\ \underset{\sim}{\upsilon} = 0 \tag{1}$$

where the dots indicate time derivative and

- $\theta = u_{i,i}$ is the volumetric deformation
- u_i : is the displacement vector
- n : is the porosity
- k_w : is the bulk modulus of water
- ϕ: is the excess pore pressure
- $\underset{\sim}{\upsilon}$: is the apparent velocity of the fluid

Taking ϕ as negative, DARCY's law is

$$\upsilon_i = \frac{k}{\gamma_w}\ \phi_{,i} \qquad\qquad \phi < 0 \tag{2}$$

where

- k : is the permeability (m/s)
- γ_w: is the specific weight of water (N/m^3)

The first field equation is then obtained by introducing (2) in (1) as follows

$$\frac{\partial\theta}{\partial t} - \frac{n}{k_w}\frac{\partial\phi}{\partial t} + \frac{k}{\gamma_w}\phi_{,ii} = 0 \tag{3}$$

The other coupled phenomen is the deformation of the soil skeleton.

Equilibrium equations are

$$\begin{array}{lll} \underset{\sim}{\nabla}\,\underset{\sim}{\sigma} + X = 0 & & \text{in } \Omega \\ \underset{\sim}{\sigma}\ \underset{\sim}{\upsilon}\,. = \underset{\sim}{t} & & \partial\Omega \end{array} \tag{4}$$

where

$\underset{\sim}{\sigma}$ is the total stress tensor
$\underset{\sim}{X}$ are the body forces
$\underset{\sim}{\nu}$ is the exterior normal to $\partial\Omega$
$\underset{\sim}{t}$ are the boundary tractions

Equations (4) can also be written, following the classical assumption of TERZAGHI, as the sum of effective $\underset{\sim}{\sigma}$ and neutral $\phi\underset{\sim}{\partial}$ pressures

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma} + \phi\underset{\sim}{\partial} \tag{5}$$

which produces

$$\begin{aligned} &\underset{\sim}{\nabla}\,\underset{\sim}{\sigma} + \underset{\sim}{X} + \underset{\sim}{\nabla}\,\phi = 0 && \text{in } \Omega \\ &\underset{\sim}{\sigma}\,\underset{\sim}{\nu} + \phi\,\underset{\sim}{\nu} = \underset{\sim}{t} && \text{in } \partial\Omega \end{aligned} \tag{6}$$

(5) can also be introduced into (3) through the use of a material law as (which necessarily needs to be expressed in effective stresses)

$$\dot{\theta} = \frac{1}{k_s}\ \dot{\sigma}'_{oct} = \frac{1}{k_s}\left(\dot{\sigma}_{oct} - \dot{\phi}\right) \tag{7}$$

where σ_{oct} is the so-called octahedral stress

$$\sigma_{oct} = \frac{1}{3}\ \sigma_{ii} \tag{8}$$

In this way equation (3) is

$$\frac{1}{k_s}\dot{\sigma}_{oct} = \left(\frac{n}{k_w} + \frac{1}{k}\right)\dot{\phi} - \frac{k}{w}\ \phi_{,ii} \tag{9}$$

or, using the same notation as VERRUIJT

$$\begin{aligned} &\gamma\,\nabla^2\phi = \dot{\phi} - \mu\ \dot{\sigma}_{oct} \\ &\alpha = \frac{kK_s}{\gamma_w} \\ &\beta = \frac{K_s n}{\gamma_w} + 1 \\ &\gamma = \frac{\alpha}{\beta}\,;\ \mu = \frac{1}{\beta} \end{aligned} \tag{10}$$

With the appropriate boundary conditions (6) and (10) defined the consolidation problem.

Boundary element discretization

In order to produce the boundary discretization it is necessary to introduce fundamental solutions of the field equations and to establish inner products in the whole domain. For equations (6) this can be done through Kelvin type solution $\underset{\sim}{u}^{*}$ in the following way

$$(\nabla \underset{\sim}{\sigma}', \underset{\sim}{u}^{*})_{\Omega} = -(\underset{\sim}{\nabla} \phi, \underset{\sim}{u}^{*})_{\Omega} - (\underset{\sim}{X}, \underset{\sim}{u}^{*})_{\Omega} \tag{11}$$

But

$$(\underset{\sim}{\nabla} \underset{\sim}{\sigma}', \underset{\sim}{u}^{*})_{\Omega} = \underset{\sim}{\nabla} (\underset{\sim}{\sigma}', \underset{\sim}{u}^{*})_{\Omega} - (\underset{\sim}{\sigma}', \underset{\sim}{\nabla} \underset{\sim}{u}^{*})_{\Omega} = (\underset{\sim}{\sigma}'\underset{\sim}{\nu}, \underset{\sim}{u}^{*})_{\partial\Omega} - (\underset{\sim}{\sigma}', \underset{\partial}{\varepsilon}^{*})_{\Omega} =$$

$$= (\underset{\sim}{\sigma}'\underset{\sim}{\nu}, \underset{\sim}{u}^{*})_{\partial\Omega} - (\underset{\sim}{\varepsilon}, \underset{\sim}{\sigma}^{*})_{\Omega} = (\underset{\sim}{\sigma}'\underset{\sim}{\nu}, \underset{\sim}{u}^{*})_{\partial\Omega} - (\underset{\sim}{u}, \underset{\sim}{t}^{*})_{\partial\Omega} \qquad -(\underset{\sim}{u}, \underset{\sim}{X}^{*})_{\Omega} \tag{12}$$

In addition

$$-(\underset{\sim}{\nabla} \phi, u^{*})_{\Omega} = -\nabla(\phi, \underset{\sim}{u}^{*})_{\Omega} + (\phi, \underset{\sim}{\nabla} \underset{\sim}{u}^{*})_{\Omega} = -(\phi \underset{\sim}{\nu}, \underset{\sim}{u}^{*})_{\partial\Omega} + (\phi, \underset{\sim}{\nabla} \underset{\sim}{u}^{*})_{\Omega} \tag{13}$$

So that

$$(\underset{\sim}{\sigma}'\underset{\sim}{\nu}, \underset{\sim}{u}^{*})_{\partial\Omega} - (\underset{\sim}{u}, \underset{\sim}{t}^{*})_{\partial\Omega} - (\underset{\sim}{u}, \underset{\sim}{X}^{*})_{\Omega} = -(\phi \underset{\sim}{\nu}, \underset{\sim}{u}^{*})_{\partial\Omega} + (\phi, \underset{\sim}{\nabla} \underset{\sim}{u}^{*})_{\Omega} - (\underset{\sim}{X}, \underset{\sim}{u}^{*})_{\Omega} \tag{14}$$

and

$$-(\underset{\sim}{u}, \underset{\sim}{X}^{*})_{\Omega} - (\underset{\sim}{u}, \underset{\sim}{t}^{*})_{\partial\Omega} = -((\underset{\sim}{\sigma}' + \phi \underset{\sim}{\partial}) \underset{\sim}{\nu}, \underset{\sim}{u}^{*})_{\partial\Omega} - (\underset{\sim}{X}, \underset{\sim}{u}^{*})_{\Omega} + (\phi, \underset{\sim}{\nabla} \underset{\sim}{u}^{*})_{\Omega} \tag{15}$$

that produces the desired relationship

$$(\underset{\sim}{u}, \underset{\sim}{X}^{*})_{\Omega} + (\underset{\sim}{u}, \underset{\sim}{t}^{*})_{\partial\Omega} = (\underset{\sim}{t}, \underset{\sim}{u}^{*})_{\partial\Omega} + (\underset{\sim}{X}, \underset{\sim}{u}^{*})_{\Omega} - (\phi, \underset{\sim}{\nabla} \underset{\sim}{u}^{*})_{\Omega} \tag{16}$$

When $\underset{\sim}{X}^{*}$ is chosen as the fundamental Kelvin solutionthe classical B.E.M. are obtained plus the additional term $(\phi, \underset{\sim}{\nabla} \underset{\sim}{u}^{*})$ which marks the coupling with the diffusion phenomena. For the usual cases $(\underset{\sim}{X}, \underset{\sim}{u})$ can be reduced to the boundary, but the coupling term has to be evaluated in volume cells.

For equation (10) and previous to any spatial discretization we introduce a time stepping by putting

$$\dot{\phi}_i \simeq \frac{1}{\Delta t}(\phi_i - \phi_{i-1}) \tag{17}$$

or any other convenient rule. In this way it is possible to write

$$\gamma \ \nabla^2 \phi - \frac{1}{\Delta t} \phi_i = - \frac{\phi_{i-1}}{\Delta t} - \mu \ \dot{\sigma}_{oct} \tag{18}$$

The operator on the right hand side is the Helmholtz one,.so that using its fundamental solution ϕ^* we can establish

$$(B \ \phi_i, \phi^*)_\Omega = (g, \ \phi^*)_\Omega \tag{19}$$

with

$$B = \gamma \ \nabla^2 - \frac{1}{\Delta t}$$

$$g = - \frac{\phi_{i-1}}{\Delta t} - \mu \ \dot{\sigma}_{oct} \tag{20}$$

arriving at the final equation

$$(\phi, g^*)_\Omega + \gamma \ (\phi, q^*)_{\partial\Omega} = \gamma \ (\phi^*, q)_{\partial\Omega} + (g, \ \phi^*)_\Omega \tag{21}$$

where q and q^* represent the fluxes associated with respectively the actual and the fundamental solution.

Computational scheme

As was indicated the solution of the problem follows a marching process progressing with the following scheme: (BANERJEE & BUTTERFIELD)

a) Assuming undrained conditions. Working on total stresses the elastic problem is solved, what allows the computation of σ_{oct}

b) Initial conditions for pore water excess are evaluated as

$$\phi^\circ = \sigma^\circ_{oct}$$

Observe that the method has no problem with management of values $\nu = \frac{1}{2}$.

c) The initial slope of the (σ_{oct}, t) curve is assumed zero, i.e.:

$$\dot{\sigma}^\circ_{oct} = 0$$

d) With $\dot{\sigma}^\circ_{oct} = 0$ (21) is solved producing the field of neutral pressures ϕ^1.

e) Knowing ϕ^1 it is an easy matter to solve (16), obtaining then the displacements $\underset{\sim}{u}^1$, the stress state $\underset{\sim}{\sigma}^1$ and the octahedral stresses σ^1_{oct}.

f) Now it is assumed that

$$\dot{\sigma}^1_{oct} = \frac{\sigma^1_{oct} - \sigma^\circ_{oct}}{\Delta t}$$

i.e.: the initial step is extrapolated from the previous computed values.

g) With σ^1_{oct} steps from d) on, can be repeated for every time interval.

The previous ideas have been implemented in a computer program whose flow-chart is described in tables 1 2 and 3. The basic support for them is the two previous programs described by PARIS et al and ROURES et al with several computational improvements. Both use linear interpolation in the boundary while the potential ϕ is assumed as having constant values inside volume cells.

As usual the most expensive part of the method is the numerical integration that in the boundary is done using a standard Gauss rule with 4 points except in certain cases where closed-form expressiones have been developed. The domain integrals are always evaluated numericals with a rule of 16 points inside each quadrilateral cell.

The integration coefficients have to be computed for each different Δt. When a unique Δt is chosen they are computed only once during the first iteration and then stored for subsequent use.

The problem is the election of the interval size in order to maintain an acceptable accuracy without increasing the cost by too much.

Roures et al suggest a three step range for decoupled problems (RENDULIC's assumption of $\dot{\sigma}_{oct} = 0 \ \forall t$) as follows

$$S = kK_{\zeta} / \gamma_w$$

$$0 < t < 0.025 \frac{L^2}{S} \quad \text{take} \quad \Delta t = 0.005 \frac{A}{S}$$

$$0.025 < t < 0.20 \frac{L^2}{S} \quad " \quad \Delta t = 0.0125 \frac{A}{S}$$

$$t > 0.20 \frac{L^2}{S} \quad " \quad \Delta t = 0.06 \frac{A}{K}$$

Nevertheless these rules have been tested only in a limited number of cases and the matter is open to further study.

Non-linear behaviour

Several improvements are easily implemented in the previous scheme at the price of growing computation time.

DAVIS and RAYMOND proposed for instance the use of variable values for the oe dometric modulus which can be done without difficulty but at the cost of computing the constants at every time step.

A better improvement is to use a non-linear elastic law to model the behaviour of the soil skeleton. (see for instance Naylor et al.)

In addition to the well-known hyperbolic laws the most interesting models are the so-called K - G, that is a

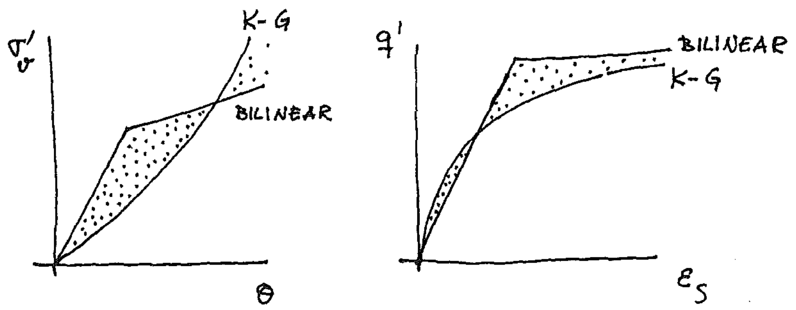

Figure 1.

continuous model, and the bilinear one.

The deviatoric and volumetric parts are separated in both models and a plasticity-type criterion is added to simulate collapse.

The K - G model allows a good representation of the increase in bulk stiffness while the bilinear model defines more neatly the collapse load.

To simulate unloading the equivalent stress inside each cell of the domain is improved through the plasticity criterion and when the limit values are reached the shear modulus is reduced to zero.

This produces good qualitative results when more accurate (but more expensive also!) analyses are not necessary.

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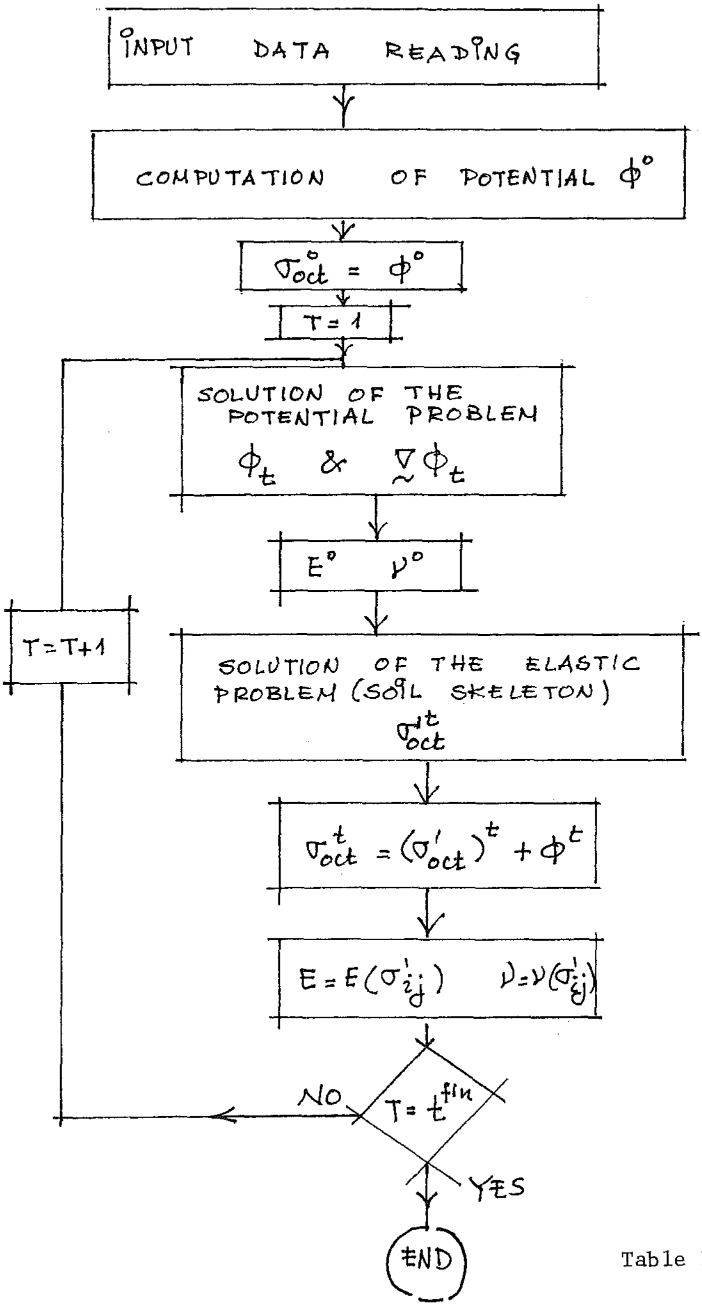
INPUT DATA READING
COMPUTATION OF POTENTIAL ϕ^{o}
$\sigma^{o}_{oct} = \phi^{o}$
$T = 1$
SOLUTION OF THE POTENTIAL PROBLEM
ϕ_t & $\underset{\sim}{\nabla} \phi_t$
E^{o} ν^{o}
SOLUTION OF THE ELASTIC PROBLEM (SOIL SKELETON)
σ'^{t}_{oct}
$\sigma^{t}_{oct} = (\sigma'_{oct})^{t} + \phi^{t}$
$E = E(\sigma'_{ij})$ $\nu = \nu(\sigma'_{ij})$
$T = t^{fin}$
NO
YES
$T = T+1$
END

Table 1.

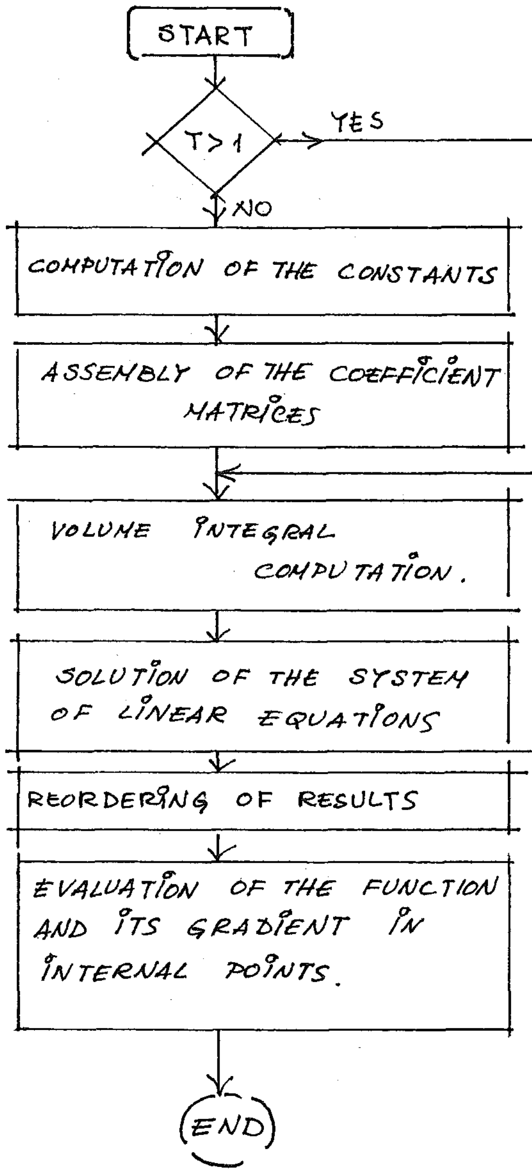

Table 2. Flow Problem Resolution

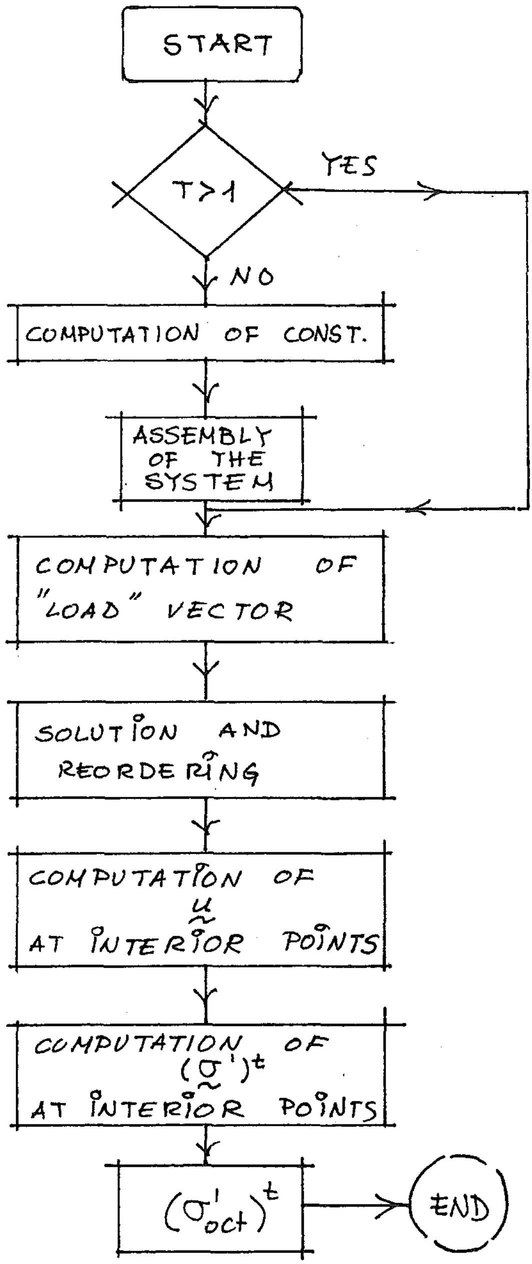

Table 3. Elastic Problem Resolution

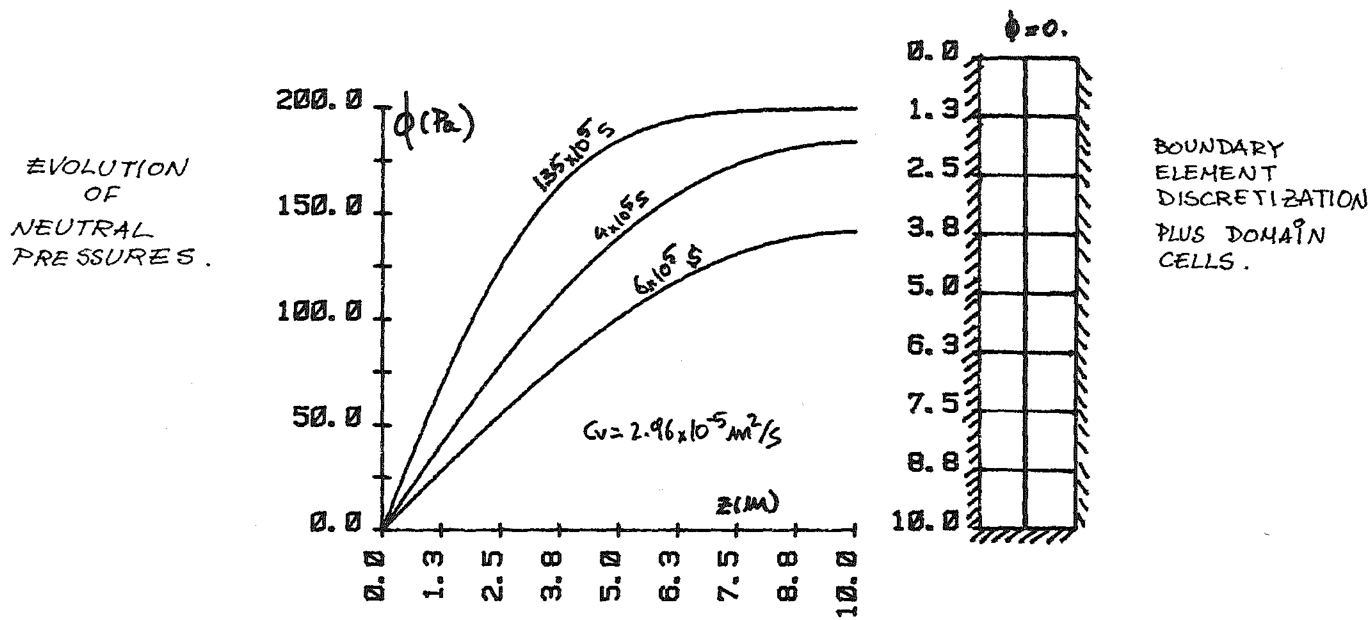

Figure 2. Uncoupled Theory Applied to One Dimensional Consolidation

$K = 1.137 \times 10^{8}$ Pa
$G = 4.0 \times 10^{7}$ Pa

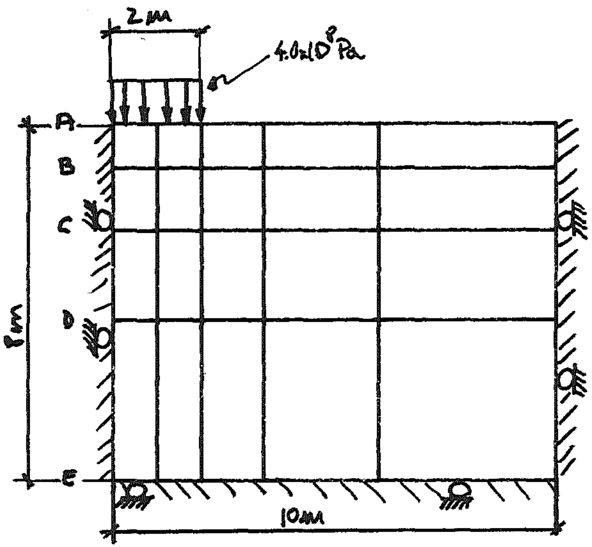

Figure 3a. Boundary Element Mesh for the Consolidation of a Layer Under a Footing Loading

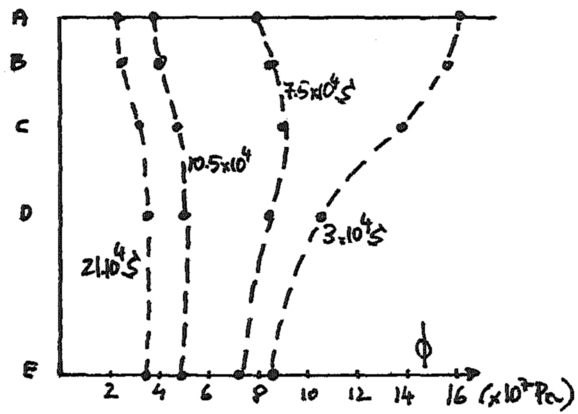

Figure 3b. Evolution of Neutral Pressure Under the Footing

APENDIX

In the solution of the elastic problem when only self weight and external applied forces are considered the following expresion is obtained for the internal displacements

$$u_j + \int_{\partial\Omega} T_{ji} u_i \, ds = \int_{\partial\Omega} U_{ji} t_i \, ds - \int_{\partial\Omega} g z U_{ji} n_i \, ds - \int_{\Omega} \zeta U_{ji,i} \, dv \tag{A1}$$

where all the terms have the usual meaning an ζ is defined as

$$\zeta = \phi - g z \tag{A2}$$

For the calculation of the internal effetive stress is necessary to derive the expresion (A1). All the derivatives can be easily performed but some care must be taken with the one corresponding to the last term in (A1) as was shown by BUI for the plasticity problem.

Let

$$I_j = \int_{\Omega} \zeta U_{ji,i} \, dv = \int_{\Omega - B_\varepsilon} \zeta U_{ji,i} \, dv + \lim_{\varepsilon \to 0} \int_{B_\varepsilon} \zeta U_{ji,i} \, dv \tag{A3}$$

$$\frac{\partial I_j}{\partial x_k} = \lim_{\varepsilon \to 0} \int_{\Omega - B_\varepsilon} \zeta \frac{\partial U_{ji,i}}{\partial x_k} \, dv - \zeta \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} U_{ji,i} \, r_{,k} \, ds \tag{A4}$$

The first integral of (A4) can be easily computed numerically and for the last term an analytical expresion can be found.